Maximum principles, boundary point lemmas and convexity

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1. Introduction.

Theorems from the title play an important role in the theory of partial differential equations, particularly in uniqueness theorems for boundary value problems for elliptic and parabolic equations. To illustrate the relationship between these theorems we consider their one-dimensional versions first.

If

$$u'' \geq 0$$

on (a, b) and u is continuous on $[a, b]$ then

A. u attains its largest value either at a or at b.
B. If u attains its largest value at $c \in (a, b)$ then u is constant.
C. If a non-constant u attains its maximum at b then $u'_-(b) > 0$ if at a then $u'_+(a) < 0$.
D. u is convex on $[a, b]$.

The one-sided derivatives in C are assumed to exist, if e.g. $u'_+(a)$ does not then instead of $u'_+(a) < 0$ one has $D^+ u(a) = \limsup\limits_{x \downarrow a} \frac{u(x) - u(a)}{x - a} < 0$.

It is common to refer to all or any of the statement $A - D$ as maximum principles.

We shall refer to A, B, C, D as the weak maximum principle, the strong maximum principle, the boundary point lemma and a convexity theorem (in that order).

Each of the theorems $A - D$ can be easily proved directly. As an illustration let us prove A. Assume, for an indirect proof, that there is a point $c \in (a,b)$ such that $u(c) > u(a), u(c) > u(b)$. Choose $\epsilon > 0$ sufficiently small to have $u(c) > u(a) + \epsilon(c-a)^2$ and $u(c) > u(b) + \epsilon(b-a)^2$ and define $v, v(x) = u(x) + \epsilon(x-a)^2$. By the Weierstrass Theorem v attains its maximum on $[a,b]$ and since $v(c) > v(a), v(c) > v(b)$, v attains its maximum at some $\xi \in (a,b)$. Then $v'(\xi) = 0$ and $v''(\xi) \leq 0$. However $v''(\xi) = u''(\xi) + 2\epsilon > 0$.

There is another way to prove all theorems A $-$ D. Firstly D is a well known theorem from calculus. Hence it suffices to prove the implications D $\Rightarrow$ C $\Rightarrow$ B $\Rightarrow$ A. For illustration let us prove D $\Rightarrow$ C. For sake of definitness let u have a maximum at b. Then $u'_-(b) \geq 0$. Assume that $u'_-(b) = 0$. Then by convexity the graph of u lies above the tangent at b which means $u(x) \geq u(b)$. Hence u is constant.

It is interesting that in some sense A $\Rightarrow$ D, namely: If for every γ the function $x \to u(x) - \gamma x$ satisfies the weak maximum principle on every interval $[\alpha, \beta] \subset [a,b]$ then u is convex on $[a,b]$. What we have said so far is true, with a grain of salt, for elliptic and parabolic inequalities in n-dimensions.

2. Elliptic inequalities.

For convenience sake we shall assume that functions denoted by letters u or v have continuous second order derivatives. The weak maximum principle can be proved for inequalities of the form

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0 \tag{1}$$

For sake of brevity the right hand side of (1) be denoted by $L(u)$. In this section we assume that L is elliptic in some set $G \subset \mathbb{R}^n$ which means that the quadratic form in λ_i, λ_j

$$\sum_{i,j=1}^{n} a_i(x) \lambda_i \lambda_j$$

is positive definite for every $x \in G$. In the statement and in the proof of the weak maximum principle for inequality (1) the interval $[a,b]$ is replaced by an open connected set G, the points a, b by the set ∂G, the boundary of G and $v(x)$ becomes $u(x)+\epsilon\sum_{i=1}^{n}(x_i-c_i)^2$. With a more sophisticated choice of v the weak maximum principle can be extended to more general operators L_1 of the forms

$$L_1(u)=L(u)+\sum_{i=1}^{n} b_i D_i u+cu,$$

with b_i, c bounded and $c\leq 0$. If c is not identically zero then one has to assume that the maximum is non-negative. More precisely: If u satisfies $L_1(u)\geq 0$, b_i are bounded, $c\leq 0$ and u attains a non-negative maximum then u reaches its maximum at some point $u\in\partial G$, the boundary of G.

The assumptions concerning the coefficients are essential as the following examples show: $u(x)=-x^4$ satisfies

$$u''(x)-b(x)u'=0$$

with $b(x)=4x^{-3}$ for $x\neq 0$ and $b(0)=0$ and u reaches a non-negative maximum on $[-1,1]$ at 0.

$u(x)=\sin x$ satisfies

$$u''+u=0$$

and its maximum on $[0,\pi]$ is not attained at one of the endpoints of this interval.

An operator L_1 is called uniformly elliptic in G if there exists a positive constant μ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j\geq\mu\sum_{i=1}^{n}\lambda_i^2$$

for all $x\in G$ and all $\lambda_i\in\mathbb{R}$. The strong maximum principle then reads: If L_1 is uniformly elliptic and its coefficients bounded, $L_1(u)\geq 0$ and u attains a non-negative maximum inside G then u is constant.

In some sense the chain of implications D $\Rightarrow$ C $\Rightarrow$ B $\Rightarrow$ A still holds but the situation is not as simple as in $\mathbb{R}$. Stronger assumption of uniform ellipticity rather than ellipticity is needed for the strong maximum principle hence to prove the best version of the weak maximum principle one has to give an independent prove of it and not merely rely on the implication B $\Rightarrow$ A. Similar situation persists with statements C, D. This is typical for the theory of partial differential equations, independent proofs for individual statements like A – D are of interest but implications like D $\Rightarrow$ C $\Rightarrow$ B $\Rightarrow$ A are often useful too.

In discussing C – D in n-dimensions we restrict ourselves to a simple case of an inequality

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} \geq 0. \tag{2}$$

The right hand side of (2) is ususally abbreviated as Δu and functions satisfying (2) are called subharmonic functions. An analogue of C for subharmonic functions then has the inquality $u'' \geq 0$ on (a, b) replaced by $\Delta u \geq 0$ in G and the corresponding statement to $u'_+(a) < 0$ or $u'_-(b) > 0$ is

$$\frac{\partial u}{\partial \ell} < 0 \tag{3}$$

at a point y at which u attains its maximum and which lies on the boundary of G. $\frac{\partial u}{\partial \ell}$ in (3) denotes a derivative in a direction ℓ pointing into G. However, in the n-dimensional case the behaviour of the boundary comes into play. Consider $G = \{(x,y); x > 0, y > 0, x^2 + y^2 < 1\}$ and $u(x,y) = -xy$. Clearly, u is subharmonic, attains its maximum at (0,0) but $\frac{\partial u}{\partial \ell} = 0$ for any direction pointing into G. There is an example (see [1]) which shows that the boundary point lemma fails for subharmonic functions in a domain in $\mathbb{R}^2$ which has its boundary formed by a smooth curve.

The condition most often used to guarantee the validity of the boundary point lemma is the so called interior sphere property. A domain G is said to have the interior sphere property if for every $y \in \partial G$ there exists an open ball $K \subset G$ and y lies on the boundary of K.

It was shown [2] that the assumption of the interior sphere property can be weaken, it is sufficient to assume that there exists a function g such that $g = 0$ on ∂G, $g > 0$ in G, grad $g \neq 0$ on ∂G and the first order partial derivatives of g are Hölder continuous.

Let us now turn our attention to D for subharmonic functions.

If u satisfies (2) in a ball $G = \{x; x \in \mathbb{R}^n, |x-a| < R\}$ and

$$M(r) = \text{Max}\{u(x); |x-a| = r\}$$

then u is a convex function of s, where $s = \log r$ if $n = 2$ and $s = \frac{1}{r^{n-2}}$ if $n > 2$.

For the proof we refer to [3] or [4], statements like this are often referred to as Hadamard's three circles theorems because if u is a modulus of an analytic function the above theorem coincides with the celebrated theorem of Hadamard from Complex Analysis. For a domain with interior sphere property the Hadamard three circles theorem for subharmonic functions implies the boundary point lemma. This was shown in [4]. Hadamard's three circles (spheres) theorems have been generalized to more general surfaces, e.g. ellipsoid's and to solutions of elliptic inequalities of the form

$$L_1(u) \geq 0.$$

In all such cases the function M becomes a convex function of some strictly increasing function s which is a solution of an ordinary differential equations. In the case of subharmonic function and $s = \log r$ this differential equation is $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0$,

3. Uniqueness of boundary value problems.

Let us show how maximum principles are employed in proofs of uniqueness theorems. We show it for the Neumann problem. A function u is a solution of the Neumann problem if it satisfies

$$\Delta u = 0$$

in some domain $G \subset I\!R^n$ is continuous in $G \cup \partial G$, and for a given function $\varphi : \partial G \to I\!R$

$$\frac{\partial u}{\partial n} = \varphi,$$

where n denotes the normal to the boundary. We prove that u is unique up to an additive constant. Let u_1 and u_2 be two solutions of the Neumann problem. Define $v = u_1 - u_2$; by the weak maximum principle v attains its maximum at some point $y \in \partial G$. If v were not a constant then by the boundary point lemma

$$\frac{\partial v}{\partial n} < 0$$

at y. However

$$\frac{\partial v}{\partial n} = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = 0.$$

Hence v must be a constant.

4. Nonlinear elliptic problems.

Since the main motivation for proving maximum principles comes from their application then in dealing with nonlinear problems it is natural to seek maximum principles for the difference of solutions. Recently Hadamard's three circles theorem (see [5]) was established for a nonlinear inequality of the form

$$f(x, u, Du(x), D^2u(x)) \geq f(x, v(x), Dv(x), D^2v(x)).$$

Du and D^2u denotes the gradient and the Hessian matrix (i.e. the matrix of the second order derivatives). To illustrate this result we again look at its greatly simplified one-dimensional version. If

(i) $f(x, u', u'') \geq f(x, v', v'')$

(ii) f satisfies a Lipschitz conditions of the form

$$|f(x,p,r) - f(x,\bar{p},.\,)| \leq L\{|p-\bar{p}| + |r-\bar{r}|\}$$

(iii) there exists a positive constant æ such that F,

$$F(t) = f(x,p,t) - æ\, t,$$

is increasing (not necessarily strictly increasing).

Then there exists a strictly increasing function z such that

$$u - v$$

is a convex function of z.

5. Further generalizations.

There are many generalizations and applications of results were discussed in the previous sections. Some of them deals with problems which are not elliptic, e.g. parabolic equations and inequalities, prime example of such equation is the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

Other generalizations consider functions which are not twice continuously differentiable. We refer the interested reader to [1] and [3]. Very little is known about maximum principles for differential inequalities of higher order but v some results are known, see e.g. [6] p. 278-282.

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